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LETTER TO THE EDITOR

Three-dimensional multiple soliton-like solutions of non-linear Klein–Gordon equations

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Abstract. A method is described which enables multiple three-dimensional soliton-like solutions to be found of non-linear Klein–Gordon equations of the type $\Box\phi = F(\phi)$. The method is extendable to coupled equations $\Box\phi = F(\phi, \psi)$; $\Box\psi = G(\phi, \psi)$.

Non-linear Klein–Gordon equations of the type

$$\Box\phi = F(\phi), \tag{1a}$$

$$\Box = \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1b}$$

play a fundamental role as model equations in theoretical physics especially if soliton solutions occur. Some important cases are the sine–Gordon equation ($F = -\sin\phi$) and the ϕ^4 equation ($F = \phi - \phi^3$) which occur in non-linear field theories (Jackiw 1977; Rajaraman 1975) and lattice dynamics (Schneider and Stoll 1975). A model for the theory of spin waves in liquid He III gives the double sine–Gordon equation where $F = \sin\phi + \frac{1}{2}\sin\frac{1}{2}\phi$ (Caudrey and Bullough 1977). Recently Gibbon *et al* (1978, to be referred to as I) showed that a correspondence exists between solutions of the above equations and that multiple soliton-like solutions of a restricted type exist.

It is shown here that there is a very simple yet important geometric significance to this correspondence which extends to the whole class of equations given in (1) yielding multiple wave solutions. More important still, this geometric property extends to coupled equations such as

$$\Box\phi = F(\phi, \psi), \qquad \Box\psi = G(\phi, \psi). \tag{2}$$

We consider surfaces in (x, y, t) space given by $g(x, y, t) = \lambda$, and then ϕ is treated as depending on the single variable g only.

The following argument revolves around a result in classical field theory which states that if

$$\Box g/(\mathbf{\Delta} g)^2 = M(g) \tag{3}$$

where $M(g)$ is an arbitrary function of g, then such surfaces are equipotential surfaces $\phi = \phi(g)$. The $\mathbf{\Delta}$ operator acts like the gradient operator (with the appropriate metric) such that $\mathbf{\Delta} \cdot \mathbf{\Delta} = \square$. The simplest choice for $\square g$ and $(\mathbf{\Delta} g)^2$ separately suggested by (3) is to take

$$\square g = A(g), \qquad (\mathbf{\Delta} g)^2 = B(g). \tag{4}$$

Equation (1) now becomes

$$B(g)\frac{d^2\phi}{dg^2} + A(g)\frac{d\phi}{dg} = F(\phi) \tag{5}$$

giving $\phi = \phi(g)$. Although A and B can be general functions of g, equation (5) is not integrable as it stands. For the moment, therefore, we shall make the choice $A = -g$; $B = -g^2$ which is appropriate for obtaining simple solutions of (4) and then return later to a more general form. Equation (5) can now be transformed directly into

$$\frac{d^2\phi}{dV^2} = -F(\phi), \qquad V = \ln g \tag{6a}$$

where

$$\square g = -g, \qquad (\Delta g)^2 = -g^2. \tag{6b}$$

Note that V is a potential function for the surfaces since $\square V = 0$. Hence, given an $F(\phi)$, as long as (6a) can be integrated to give $\phi = \phi(\ln g)$, then solutions in (x, y, t) space are determined by solutions of the pair of equations (6b). One set of multiple wave solutions is

$$g = \sum_{i=1}^{N} \exp \theta_i \tag{7a}$$

$$\theta_i = p_i x + q_i y - \omega_i t + \delta_i \tag{7b}$$

$$p_i^2 + q_i^2 - \omega_i^2 = 1 \tag{7c}$$

with the ${}^N C_2$ conditions on the motion

$$(p_i - p_j)^2 + (q_i - q_j)^2 - (\omega_i - \omega_j)^2 = 0. \tag{7d}$$

Equation (7c) expresses the mass–energy relation for the ith wave. The restrictions given in (7d) show that two 'particles' of unit mass with momentum components (ω_i, p_i, q_i) and (ω_j, p_j, q_j) give rise to a third with components $(\omega_i - \omega_j, p_i - p_j, q_i - q_j)$ which has *zero* mass. Other solutions of (6b) were given in I. There are restrictions on the number of waves allowed since equations (7c, d) are overdetermined when $N > 5$.

Example 1. ϕ^4 equation: $F(\phi) = \frac{1}{2}(\phi - \phi^3)$. The well known kink-type solution of (1) is

$$\phi = \tanh \tfrac{1}{2}V = [(g-1)/(g+1)]. \tag{8}$$

Example 2. $F(\phi) = \phi^3 - \phi$. A solution for boundary conditions $\phi \to 0$; $x \to \pm\infty$ is

$$\phi = \frac{\exp V}{1 + \frac{1}{8}\exp 2V} = \frac{g}{1 + \frac{1}{8}g^2}. \tag{9}$$

A plot of ϕ against x and y (at $t = 0$) is given in figure 1 for three waves. These solutions are not full N soliton solutions in their truest sense since the wavefronts do not extend to infinity but they are nevertheless multiple soliton plane wave solutions which can interact at different speeds and angles. Results for the double sine–Gordon

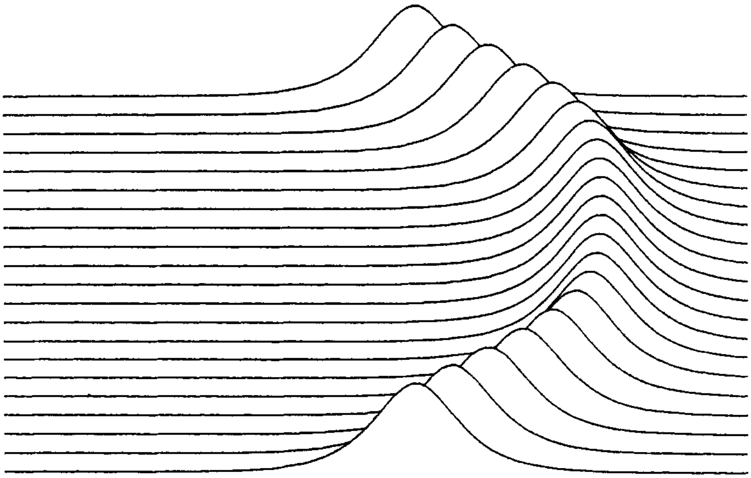

Figure 1. Plot of ϕ against x (continuous) and y (discrete) at $t=0$ for three waves: $p_1=q_1=\omega_1=1$; $p_2=1$, $q_2=-\omega_2=-1$; $p_3=1$, $q_3=\omega_3=0$.

equation were found in I by a heuristic method and will not be repeated here. It is possible to generalise (5) under the assumption that $A=\frac{1}{2}B'$. Equation (5) is still integrable and yields a transformation between $B(g)$ and ϕ:

$$\int B^{-1/2}\,\mathrm{d}g=\int\left(2\int F\,\mathrm{d}\phi\right)^{-1/2}\mathrm{d}\phi. \tag{10}$$

However it is then possible to do the same procedure over again and reduce

$$\Box g=\tfrac{1}{2}B', \qquad (\boldsymbol{\Delta} g)^2=B \tag{11a}$$

to the pair given in (6*b*)

$$\Box g'=-g', \qquad (\boldsymbol{\Delta} g')^2=-g'^2 \tag{11b}$$

by calculating a transformation $g=g(g')$.

Hence (1) can be transformed either directly to (6*b*) with $V=\ln g$ or if necessary to some intermediate pair (11*a*).

The great value of this procedure is shown when considering coupled wave equations in two scalar variables as in (2). Consider therefore two sets of equipotential surfaces $g(x,y,t)=\lambda_1$; $h(x,y,t)=\lambda_2$ with two sets of coupled equations analogous to (6*b*)

$$\begin{aligned}\Box g&=-g, \qquad (\boldsymbol{\Delta} g)^2=-g^2;\\ \Box h&=-h, \qquad (\boldsymbol{\Delta} h)^2=-h^2,\end{aligned} \tag{12}$$

and $(\boldsymbol{\Delta} g)\cdot(\boldsymbol{\Delta} h)=-gh$.

Following the same procedure which produced (6*a*), a similar result is obtained. Equation (2) becomes

$$\begin{aligned}&(\partial^2\phi/\partial\zeta^2)_\xi=-F(\phi,\psi), \qquad (\partial^2\psi/\partial\zeta^2)_\xi=-G(\phi,\psi)\\ &\zeta=\tfrac{1}{2}(V_1+V_2)=\tfrac{1}{2}\ln(gh)\\ &\xi=\tfrac{1}{2}(V_1-V_2)=\tfrac{1}{2}\ln(g/h)\end{aligned} \tag{13}$$

where two potentials $V_1 = \ln g$; $V_2 = \ln h$ have been introduced. The variable ξ, although redundant in (13), can still reappear as an integration constant. One example of this type is

$$\Box\phi = \phi^2\psi - \phi, \tag{14a}$$

$$\Box\psi = \psi^2\phi - \psi. \tag{14b}$$

Through the transformations $\phi = \alpha\Phi$; $\psi = \alpha^{-1}\Phi$, (14*a*, *b*) reduce to

$$\Phi_{\zeta\zeta} = \Phi - \Phi^3. \tag{15}$$

For solutions which have boundary conditions $\Phi \to 0$; $x \to \pm\infty$, we have

$$\Phi = (\exp \zeta)/(1 + \tfrac{1}{8} \exp 2\zeta). \tag{16}$$

Although the constant α can be any arbitrary function of ξ, we shall for convenience choose it to be $\alpha = \exp \xi$ giving

$$\phi = g/(1 + \tfrac{1}{8}gh), \qquad \psi = h/(1 + \tfrac{1}{8}gh). \tag{17}$$

Solutions of (17) are interesting since they can display some greater generality than (7). One set has the same relationships between the (p_i, q_i, ω_i) as in (7):

$$g = \sum_{i=1}^{N} \exp \theta_i, \qquad h = \sum_{i=1}^{M} \exp (\theta_i + \Delta). \tag{18}$$

Note that the phases of the two parts can be different and also g and h do not have to have the same number of exponentials. The coupled equations (14*a*, *b*) also contain as a special case $\psi = \phi^*$ ($h = g^*$) for complex valued fields.

As long as the necessary integrals in (13) can be performed, more general coupled equations can be solved. If the metric in the d'Alembertian is Euclidean then the situation changes and solutions may not exist. For instance Hilbert's theorem (Willmore 1959) does not allow solutions for surfaces which have constant negative curvature in three-dimensional Euclidean space.

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